1- Functions of Several Variables

(1)Find the first partial derivatives of the following functions :

$(a)\ f(x,y) = x^2 + y^4 + xy + 3$ $(b)\ f(x,y) = 3^x + xy^4 + \ln y$

$(c)\ f(x,y) = \cosh x^3 + y\sin y$ $(d)\ f(x,y) = x\sin x + 3\sinh 2y$

$(e)\ f(x,y,z) = x^2 + z\,y^4 + x\,e^{2z}$ $(f)\ f(x,y,z) = 2x^4 + \ln(3y + z^3)$

$(g)\ f(x,y,z) = y^{-3} + \sin(x^2 + z)$ $(h)\ f(x,y,z) = x\,e^x + 3z - \sinh(yz)$

(2)Find f_{xx} , f_{yy} , f_{xy} , f_{yx} for the following functions :

$(a)\ f(x,y) = x^3 + y^4 + xy + 3$ $(b)\ f(x,y) = 3^x + xy^4 + \ln y$

$(c)\ f(x,y) = \cosh x^3 + y\sin y$ $(d)\ f(x,y) = x\sinh x + 3\cos 2y$

$(e)\ f(x,y) = x^2 + y^4 + x\,e^{2y}$ $(f)\ f(x,y) = 2x^4 + \ln(3 + y^3)$

(3)For the following functions, show that : $f_{xx} + f_{yy} = 0$

$(a)\ f(x,y) = x + e^y\cos x$ $(b)\ f(x,y) = x^2 - y^2$ $(c)\ f(x,y) = \tan^{-1}\frac{y}{x}$

(4)Find $\frac{dy}{dx}$ from the following equations :

$(a)\ x^3 + y^4 + x\tanh y = 3$ $(b)\ 3^x\sin y + xy^4 + \ln y = 0$

$(c)\ y\cosh x^3 + x\sin y = 8$ $(d)\ x\sin x + y\cos y = 0$

$(e)\ x^2 + y^4 + x\,e^y + 4 = 0$ $(f)\ 2x^4 + \ln(x + y^3) - 2y = 0$

(5)Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ from the following equations :

$(a)\ x^3 + e^{yz} + x\,y\sin z = 3$ $(b)\ 3^x\sin y + xy^4 + z\ln z = 0$

$(c)\ y\cosh x + x\sin y = z\sinh z$ $(d)\ x\tan y + y\cos x + ze^z = 0$

$(e)\ x^2 + y^4 + z\sin(xyz) = 0$ $(f)\ 2x^4 + y + 2z + \ln(xyz) = 0$

(6)Verify Euler's theorem for the following functions :

$(a)\ f(x,y) = x^3 + y^3 + x\,y^2$ $(b)\ f(x,y) = 3x^4 + x\,y^3 - 2y^4$

$(c)\ f(x,y) = \ln x - \ln y$ $(d)\ f(x,y) = 5x + \sqrt{x^2 + y^2}$

$(e)\ f(x,y,z) = x^4 + y^4 + z\,y^3$ $(f)\ f(x,y,z) = 2x^2 + yz + 3z^2$

$(g)\ f(x,y,z) = x + 2z + \dfrac{y^2}{x + z}$ $(h)\ f(x,y,z) = \sin\dfrac{x}{y+z} - \cos\dfrac{y}{z}$

(7) If $u = \frac{x}{y} + \sin\frac{x}{y+z} - sinh\frac{y}{z}$. Show that : $x\,u_x + y\,u_y + z\,u_z = 0$

(8) If $u = \ln\frac{x^3-y^3}{x^2+y^2}$. Show that : $x\,u_x + y\,u_y = 1$

(9) If $u = \tan^{-1}\frac{x^3+y^3}{x+y}$. Show that : $x\,u_x + y\,u_y = \sin 2u$

(10)Find the envelope of the following curves :

$(a)\ x^2 + (y-a)^2 = 2a$ $(b)\ (x-b)^2 + y^2 = 4b$

$(c)\ (x-c)^2 + (y-c)^2 = 8$ $(d)\ (x-\cos\alpha)^2 + (y-\sin\alpha)^2 = 1$

$(e)\ x\cos\theta + y\sin\theta = 2$ $(f)\ x\sin\beta + y\cos\beta = 4$

(11)Find the extrema of the following functions:

$(a)\ f(x,y) = 2x + 3y$ $(b)\ f(x,y) = x^2 + y^3 - 4xy + 4y$

$(c)\ f(x,y) = 3 + \ln x - \ln y$ $(d)\ f(x,y) = 1 - x^2 - y^2 - 4x + 2y$

$(e)\ f(x,y) = x^3 - y^3 + 3xy$ $(f)\ f(x,y) = x^2 + y^2 - 4x + 6y + 2$

$(g)\ f(x,y) = x^2 + 3y^2 + 2xy$ $(h)\ f(x,y) = x^4 + y^3 + 32x - 9y$

(12)Find the extrema of the following functions subject to the given constraints :

$(a)\ f(x,y) = xy$; $4x^2 + y^2 = 8$

$(b)\ f(x,y) = x + y + z$; $2x + 3y = 2$

$(c)\ f(x,y) = x + y + z$; $x^2 + y^2 + z^2 - 27 = 0$

$(d)\ f(x,y,z) = x^2 + y^2 + z^2$; $x + y + z = 12$

$(e)\ f(x,y,z) = x^2 + y^2 + z^2$; $x - y + z = 6$

(13)Find the point on the plane x + 2y – z = 10 nearest to the point (1, 2, 3).

(14)Find the volume of the largest rectangular box that can be inscribed in the ellipsoid : $16x^2 + 4y^2 + 9z^2 = 144$

2- Vectors Analysis

(1) If $\overline{U} = 2i - 2j + k,\ \overline{V} = 3i + 4k,\ \overline{W} = i - 2j + 2k$. Find :

$(a)\ \overline{U} + \overline{V}$ $(b)\ \overline{U} + \overline{V} + \overline{W}$ $(c)\ \overline{U} + 3\overline{V} - 2\overline{W}$

$(d)\ \overline{U}.\overline{V}$ $(e)\ \overline{U}.\overline{V} + \overline{V}.\overline{W}$ $(f)\ \overline{U}x\overline{V} + \overline{V}x\overline{W}$

$(g)\ |\overline{U}| + |\overline{V}|$ $(h)\ |\overline{U}| + \overline{V}.\overline{W}$ $(i)\ |\overline{U}x\overline{V}| - |\overline{V}x\overline{W}|$

(2)Find the angle between the vectors : $\overline{U} = 2i - j + 2k$ and $\overline{V} = 3i - 4j$

(3)Let P(1, 2, – 1), Q(3, 0, – 3) and S(5, 4, 1) be vertices of a triangle.
Find the interior angles of this triangle.

(4) Find the value of the constant c so that the vectors $\overline{U} = 3i - j + ck$ and
$\overline{V} = i - 2j + 2k$ are perpendicular.

(5)Find a unit vector perpendicular to the plane of the two vectors :
$\overline{U} = 2i - j + 3k$ and $\overline{V} = -i + 4j$.

(6) If $\overline{U} = (t^2 + 2t)i + (t\sin t)\,j + (2 + \cos t)k$. Find $\frac{d\overline{U}}{dt}, \frac{d^2\overline{U}}{dt^2}$

(7) If $\overline{U} = (t + e^t)i + (t + \sinh t)\,j + (3t + \cos t)k$. Find $\frac{d\overline{U}}{dt}, \left|\frac{d^2\overline{U}}{dt^2}\right|$ at t = 0.

(8) If $\overline{U} = (x^2 + yz)i + (y\sin x)\,j + (2^y + \cos z)k$.
Find $\frac{\partial\overline{U}}{\partial x}, \frac{\partial\overline{U}}{\partial y}, \frac{\partial\overline{U}}{\partial z}, \overline{U}_{xx}, \overline{U}_{yy}, \overline{U}_{zz}, \overline{U}_{xy}, \overline{U}_{xz}, \overline{U}_{yz}$

(9) Find $\nabla\varphi$ where :

$(a)\ \varphi = x^4 + yz + 2^{yz}$ $(b)\ \varphi = y\sin x + z\cos y + e^z$ $(c)\ \varphi = x^3 + xyz + e^{xyz}$

(10) If $\varphi = x^4\sin y + e^z$, $\psi = e^{xyz}$. Find $\nabla\varphi$, $\nabla\psi$ and $\nabla(\phi + \psi)$.

(11)Find the angle between the normal vectors to the surfaces : $x^2 + y^2 + z^2 = 16$
and $x^2 + y^2 - 3z + 2 = 0$ at the point $(2, \sqrt{3}, 3)$.

(12)Find $\nabla.\overline{U}$ where :

(a) $\overline{U} = (x^2y)i + (yz)j + \cos(xz)k$.

(b) $\overline{U} = (e^xy)i - (y + z)j + \sin(2x + 3z)k$.

(c) $\overline{U} = (yz)i + (xyz)j + \ln(x + 3y + 2z)k$.

(13)Find $\nabla.\overline{U}$ at the point (1, 1, 1) where : $\overline{U} = (yz)i + (xy + z)j + \ln(xyz)k$.

(14)Find $\nabla x \overline{U}$ where :

(a) $\overline{U} = (x^2y)i + (yz)j + \cos(xz)k.$

(b) $\overline{U} = (e^x y)i - (y+z)j + \sin(2x+3z)k.$

(c) $\overline{U} = (yz)i + (xyz)j + \ln(x+3y+2z)k.$

(15)Find $\nabla x \overline{U}$ at the point (1, 1, 1) where : $\overline{U} = (yz)i + (xy+z)j + \ln(xyz)k.$

(16)Find $\nabla.(\overline{U}+\overline{V})$, $\nabla x(\overline{U}+\overline{V})$, $\nabla.(\overline{U}x\overline{V})$, $\nabla(\overline{U}.\overline{V})$ where:

$\overline{U} = x^2\, i + xy\, j + y^3z\, k$ and $\overline{V} = (yz)i - y\, j + xz\, k.$

(17) Find the integral : $\int_{(0,0)}^{(1,1)} (x+y)dx + (2x-3y)dy$ along the curves :

(a) y = x (b) $x^2 = y$ (c) $x = y^2$ (d) $y = x^3$

(18) Find the integral : $\int_{(0,0)}^{(1,1)} (x+2y)dx + (2x-y)dy$ along the curves :

(a) y = x (b) $x^2 = y$ (c) $x = y^2$ (d) $y = x^3$

(19) Find the integral : $\oint_C (xy) + (x+y)dy$ along the curve C given by :

(a) $x^2 = y$, $x = y^2$ (b) $x^2 + y^2 = 4$

(20) Find the integral : $\oint_C (x+2y) + (x^2-y)dy$ where the curve C is the sides of the triangle of vertices : (0, 0), (2, 0), (2, 2).

(21) Find the integral : $\oint_C (2xy) + (x^2+y^3)dy$ where the curve C given by :

(a) $y = x$, $y = x^2$ (b) $x^2 + y^2 = 5$

(22) Find the integral : $\oint_C (3x^2y) + (x^3-2y)dy$ where the curve C is the sides of the rectangle of vertices : (0, 0), (2, 0), (2, 2), (0. 2).

3- Complex Functions

(1)Determine which of the following functions are harmonic :

$(a)\ u = x\sin y - y\cos x$ $(b)\ u = x^2 - y^2$ $(c)\ u = x^2 + 2y - y^2 + 2$

$(d)\ v = x^3 - 3xy^2 + x$ $(e)\ v = y^2 + xy - x^2$ $(f)\ v = x^2 + 2x - y^2 - 3$

(2)Show that the following functions satisfy Riemman equations :

$(a)\ f(z) = 2z + 3$ $(b)\ f(z) = z^2 - 3z$ $(c)\ f(z) = z + e^z$

$(d)\ f(z) = 3 + \sin 2z$ $(e)\ f(z) = z - \cosh z$ $(f)\ f(z) = z^2 + \cos z$

(3)Find the residues of the following functions:

$(a)\ f(z) = 2z + 3$ $(b)\ f(z) = \frac{z-1}{z^2(z-2)}$ $(c)\ f(z) = \frac{\sin z}{z}$

$(d)\ f(z) = \frac{e^z}{z^3(z-1)}$ $(e)\ f(z) = \frac{e^z}{z-i}$ $(f)\ f(z) = \frac{z}{z^2 - 5z + 6}$

(4)Show that : If c is the ellipse z(t) = 5cost + i4sint. Then

(a) $\int_c \frac{1}{z+8}dz = 0$ (b) $\int_c \frac{e^{3z}}{z-3\pi i}dz = 0$ (c) $\int_c \frac{\cosh 2z}{z-8i}dz = 0$

(d) $\int_c \frac{\ln(z+6)}{z^2+36}dz = 0$ (e) $\int_c \frac{\sin 4z}{z^2-36}dz = 0$ (f) $\int_c \frac{\sinh z}{z-3\pi i}dz = 0$

(5)Show that : If C is the circle |z| =1. Then

(a) $\int_c \frac{1}{z}dz = 2\pi i$ (b) $\int_c \frac{1}{4z+i}dz = \frac{\pi i}{2}$ (c) $\int_c \frac{\cos z}{z}dz = 2\pi i$

(6)Show that : If C is the circle |z| = 4. Then

(a) $\int_c \frac{z}{z^2-1}dz = 2\pi i$ (b) $\int_c \frac{z+1}{z^2(z+2)}dz = 0$ (c) $\int_c \frac{z^2}{(z^2+3z+2)^2}dz = 0$

(d) $\int_c \frac{1}{z^2+z+1}dz = 0$ (e) $\int_c \frac{1}{z(z-2)^3}dz = 0$

(7)Show that : If C is the circle |z| = 3. Then

(a) $\int_c \frac{z+2}{z(z+1)}dz = 2\pi i$ (b) $\int_c \frac{1}{z(z+2)}dz = 0$ (c) $\int_c \frac{1}{(z+1)^3}dz = 0$

(d) $\int_c \frac{1}{z(z+1)^2}dz = 0$ (e) $\int_c \frac{z}{(z+1)(z+2)}dz = 2\pi i$

(8)Show that :

(a) $\int_0^{2\pi} \frac{d\theta}{10-6\sin\theta} = \frac{\pi}{4}$ (b) $\int_0^{2\pi} \frac{d\theta}{3+\cos\theta+2\sin\theta} = \pi$

(c) $\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}d\theta = \frac{\pi}{4}$ (d) $\int_0^{2\pi} \frac{1}{(5-3\sin\theta)^2}d\theta = \frac{5\pi}{32}$

(e) $\int_{-\infty}^{\infty} \frac{x^2}{1+x^6}dx = \frac{\pi}{3}$ (f) $\int_{-\infty}^{\infty} \frac{\cos 2x}{(9+x^2)^2}dx = \frac{7\pi}{108e^6}$

4- Fourier Series

(1)Find the Fourier series of the following functions :

$(a) f(x) = 4 - x^2, \quad -2 \le x \le 2, \quad f(x+4) = f(x)$

$(b) f(x) = x^2, \quad -2 \le x \le 2, \quad f(x+4) = f(x)$

$(c) f(x) = x + 1, \quad -1 \le x \le 1, \quad f(x+2) = f(x)$

$(d) f(x) = x \sin x, \quad -\pi \le x \le \pi, \quad f(x+2\pi) = f(x)$

$(e) f(x) = |x|, \quad -3 \le x \le 3, \quad f(x+6) = f(x)$

$(f) f(x) = |\sin x|, \quad -\pi \le x \le \pi, \quad f(x+2\pi) = f(x)$

$(g) f(x) = |x| - 1, \quad -1 \le x \le 1, \quad f(x+2) = f(x)$

$(h) f(x) = \begin{cases} -1, & -1 \le x \le 0 \\ 1 & 0 < x \le 1 \end{cases} \quad f(x+2) = f(x)$

$(i) f(x) = \begin{cases} 1 + x, & -1 \le x \le 0 \\ 1 - x, & 0 < x \le 1 \end{cases} \quad f(x+2) = f(x)$

$(j) f(x) = \begin{cases} 0, & -2 \le x \le 0 \\ x^2, & 0 < x \le 2 \end{cases} \quad f(x+4) = f(x)$

$(k) f(x) = \begin{cases} 0, & -2 \le x \le 0 \\ 1, & 0 < x \le 2 \end{cases} \quad f(x+4) = f(x)$

$(l) f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin x, & 0 < x \le \pi \end{cases} \quad f(x+2\pi) = f(x)$

$(m) f(x) = sin^5 x, \quad 0 \le x \le 2\pi, \quad f(x+2\pi) = f(x)$

(2) If $f(x) = x + 1$, x in [0, 1], $f(x + 2) = f(x)$

Find: (a) Fourier sine series (b) Fourier cosine series

(3) $f(x) = x^2, \quad 0 \le x \le 2, \quad f(x+4) = f(x)$

Find: (a) Fourier sine series (b) Fourier cosine series

(4) $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases} \quad f(x+4) = f(x)$

Find: (a) Fourier sine series (b) Fourier cosine series

(5)Using the Fourier series of the function of problem (1.d) to find the sum :

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2-1} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} \dots$$

(6)Using the Fourier series of the function of problem (1.k) to find the sum :

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \dots$$